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NUMERICAL ANALYSIS OF A STEFAN PROBLEM

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ABSTRACT

As a typical free boundary problem, a Stefan problem is studied from two analytical and numerical points of view. In the first one, by changing the dependent variable which stands for the temperature distribution, the Stefan problem is transformed into a variational inequality (V.I.). It is well known that V.I. can be approximated by a penalized problem. The second one is the method of the integrated penalty which gives a new interpretation of this penalized problem. For these different problems, existence and convergence theorems are given and, moreover, numerical methods to solve them are presented. Finally some numerical results are given.

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SIGNIFICANCE AND EXPLANATION

We can observe many phenomena involving Free Boundaries in various fields of engineering and applied sciences, for example, free boundary problems in optimum design, the pollution of air and water, the equilibrium of plasma. For such problems it is important to develop reliable computational methods which are practically efficient in applications. Naturally, a crucial point in numerical methods for free boundary problem is how to deal with the moving boundary. In order to meet this difficulty, various approaches have been proposed; they can be classified into two groups. One of them is to follow the free boundary directly. The other is to tranform the original problem into an auxiliary problem with a fixed boundary. ~~Our~~ technique uses the penalty method and falls into the second group.

In this paper we apply this method to the Stefan problem which arises in the analysis of melting of ice adjacent to a heated body of water. The essential idea of ~~our~~ method is to transform the Stefan problem into an initial-boundary value problem for the heat equation defined in a cylindrical domain, occupied jointly by water and ice, with an artificial heat absorbtion in the ice region. ~~Our~~ formulation is closely related to the approach using variational inequalities.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

NUMERICAL ANALYSIS OF A STEFAN PROBLEM

H. Kawarada* and C. Saguez**

INTRODUCTION

Free boundary problems appear in various fields of engineering and applied sciences, for example, problems in mechanics of continuous media, the equilibrium of plasma, the pollution of air and water and others. Here we restrict our interest to systems of Stefan type where exists a change of phase (solidification, liquefaction, sublimation...)

So, we consider a problem of Stefan type $(Pr)_0$ in one dimensional space. For this problem, we introduce successively i) a variational inequality (V.I.) by changing the depending variable of $(Pr)_0$; ii) a first

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penalized problem $(Pr)_1$, which is directly introduced from $(Pr)_0$ by means of the method of integrated penalty; iii) a second penalized problem $(Pr)_2$ by changing the depending variable of $(Pr)_1$, which arises as a penalized problem associated with (V.I.). The plan is following:

1. FORMULATION OF THE PROBLEM AND V.I.

We consider a one phase Stefan problem in one dimension. $\theta(x,t)$ denotes the temperature of the solid and $x = s(t)$ is the equation of the free boundary. For simplicity, we take all physical constants equal to 1.

The problem $(Pr)_0$ is to find $\{\theta(x,t),\ x = s(t)\}$ such that:

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2} \qquad 0 < x < s(t), \quad 0 < t < T, \tag{1.1}$$

$$\theta(0,t) = f(t) \qquad 0 < t < T, \tag{1.2}$$

$$\theta(s(t),t) = 0 \qquad 0 < t < T, \tag{1.3}$$

$$\theta(x,0) = \theta_0(x) \qquad 0 < x < b, \tag{1.4}$$

$$\frac{ds}{dt} = -\frac{\partial\theta}{\partial x}(s(t),t) + h(t) \qquad 0 < t < T, \tag{1.5}$$

$$(1.6) \qquad s(0) = b,$$

where $f(t)$, $h(t)$, $\theta_0(x)$ and b are given. The term $h(t)$ in (1.5) can represent, for example, a heat source along the free boundary. Such a phenomenon can be observed on the heat shield of a space capsule during the entry phase in the atmosphere or in the continuous casting process taking the convection in the liquid into account [8].

Here we assume:

$$(\mathrm{A1}) \begin{cases} f(t) \in C^{1,\tau}(0,T)^* \quad (0<\tau<1), \quad 0 \le f(t) \le \alpha \text{ in } [0,T], \\ h(t) \in C^1(0,T), \quad 0 \le h(t) \le \beta^{**} \quad \text{in } [0,T], \\ 0 < b < +\infty, \\ \theta_0(x) \in C^2(0,b), \quad \theta_0(b) = 0, \quad 0 \le \theta_0(x) \le \gamma \text{ in } [0,b], \\ f(0) = \theta_0(0). \end{cases}$$

Let R_0 be such that $s(t) < R_0$ for any $t \in (0,T)$. We denote by $\tilde{\theta}(x,t)$ the extension of $\theta(x,t)$ by zero on the region $(s(t) < x < R_0,\ 0 < t < T)$. We introduce the new variable $y(x,t)$:

$$(1.7) \qquad y(x,t) = \int_x^{R_0} \tilde{\theta}(\xi,t)\,d\xi.$$

---

* $C^{m,\tau}(0,T)$, m integer, $0<\tau<1$, is the space of m times continuously differentiable functions in $(0,T)$, such that the m th derivatives are Hölder continuous with exponent τ.

** The condition $h(t) \ge 0$ can be relaxed.

Noting that, by the maximum principle, $\tilde{\theta}(x,t) \geq 0$, we deduce:

$$(1.8) \qquad -\frac{ds}{dt} + h(t) \leq 0 \qquad \text{in } 0 \leq t \leq T.$$

Then, after some calculations, we prove that y is solution of the following variational inequality (V.I.): ([11] C. Saguez)

To find
$$\begin{cases} y \in L^2(0,T\,;\,H^1(\Omega)), \\ \frac{\partial y}{\partial t} \in L^2(0,T\,;\,L^2(\Omega)), \\ s(t) \in H^1(0,T), \\ \Omega = (0,R_0) \end{cases}$$

such that

$$(1.9) \qquad \left(\frac{\partial y}{\partial t}, \phi - y\right) + a(y, \phi - y) \geq \left(-\frac{ds}{dt} + h(t), \phi - y\right) + f(t)(\phi(0) - y(0,t)),$$

$$\phi \in K = \{\phi \in H^1(\Omega)\,;\, \phi(R) = 0,\ \phi \geq 0 \text{ a.e. in } \Omega\},$$

$$(1.10) \qquad y(x,0) = y_0(x) = \int_x^{R_0} \tilde{\theta}_0(\xi)\,d\xi,$$

$$(1.11) \qquad \begin{cases} s(t) = \text{Inf}\,\{x \mid y(x,t) = 0\}, & 0 < t < T, \\ s(0) = b \end{cases}$$

$$(1.12) \qquad s(t) < R_0$$

where $(u,v) = \int_0^{R_0} u(x)v(x)\,dx$, $a(u,v) = \int_0^{R_0} \frac{du}{dx} \cdot \frac{dv}{dx}\,dx$

and
$$\tilde{\theta}_0(x) = \begin{cases} \theta_0(x) & 0 \leq x \leq b, \\ 0 & b \leq x \leq R_0. \end{cases}$$

This inequality differs from an ordinary variational inequality by the presence of the term $-\frac{ds}{dt} + h(t)$ in the right member of (1.9), together with $s(t)$ which is defined in (1.11). An inequality of similar type was studied by A. Friedman-D. Kinderlehrer [3] and Nguyen-Din-Tri [10]. A complete study of (1.9)-(1.12) is presented by C. Saguez [11].

2. INTRODUCTION OF THE PENALIZED PROBLEMS $(Pr)_1$ AND $(Pr)_2$

2.1 The method of integrated penalty

Here we give the principle of the method of integrated penalty introduced in [6] by one of the author. Originally, the penalty method has been widely used in optimization problems with constraints. It is also a convenient tool for the study of partial differential equations (see for example J.L. Lions [9], H. Fujita and N. Sauer [4]).

Now let us consider the following penalized problem defined in $Q_T = (0,R_1) \times (0,T)$:

for any $\varepsilon > 0$,

$$\frac{\partial \pi^\varepsilon}{\partial t} = \frac{\partial^2 \pi^\varepsilon}{\partial x^2} - \frac{1}{\varepsilon}\chi(x,t)\pi^\varepsilon \qquad \text{in } Q_T, \tag{2.1}$$

$$\pi^\varepsilon(0,t) = f(t) \qquad 0 < t < T, \tag{2.2}$$

$$\pi^\varepsilon(R_1,t) = 0 \qquad 0 < t < T, \tag{2.3}$$

$$\pi^\varepsilon(x,0) = \gamma_0(x) \qquad 0 < x < R_1, \tag{2.4}$$

where $\chi = \chi(x,t)$ in (2.1) is characterized by the moving boundary $x = \psi(t)$ such that:

$$\chi(x,t) = \begin{cases} 0 & \text{in } Q_T^{(0)} = \{(x,t)\,;\, 0 < x \leq \psi(t),\ 0 < t < T\} \\ 1 & \text{in } Q_T^{(1)} = Q_T - Q_T^{(0)} \end{cases}$$

We assume that $x = \psi(t)$ is sufficiently smooth in $(0,T)$, $0 < \psi(t) < R_1$ $(0 < t \leq T)$ and $\psi(0) = b$. The problem (2.1)-(2.4) has a solution $\pi^\varepsilon \in H^{2,1}(Q_T)$*.

We easily verify that as $\varepsilon \to 0$, π^ε converges to a function π^0 strongly in $H^1(Q_T^{(0)})$ and converges to zero strongly in $H^1(Q_T^{(1)})$**. In fact, $\pi^0 \in C^{2,1}(Q_T^{(0)})$ is the unique solution of the following initial boundary value problem:

$$\frac{\partial \pi^0}{\partial t} = \frac{\partial^2 \pi^0}{\partial x^2} \qquad \text{in } Q_T^{(0)} \tag{2.5}$$

$$\pi^{(0)}(0,t) = f(t) \qquad 0 < t < T, \tag{2.6}$$

$$\pi^0(\psi(t),t) = 0 \qquad 0 < t < T, \tag{2.7}$$

$$\pi^0(x,0) = \theta_0(x) \qquad 0 < x < b. \tag{2.8}$$

We define:

$$p_\varepsilon = p_\varepsilon(x,t) = \left(\frac{1}{\varepsilon}\chi\,\pi^\varepsilon\right)(x,t) \qquad \text{in } Q_T, \tag{2.9}$$

---

* $H^{m,n}(Q_T)$, m, n, integer, is the Sobolev space:

$$\{\phi \in L^2(Q_T),\ D_x^\mu \phi \in L^2(Q_T)\ \ (0 \leq \mu \leq m),$$
$$D_t^\gamma \phi \in L^2(Q_T)\ \ (0 \leq \gamma \leq n)\}.$$

** $H^1(Q_T) \equiv H^{1,1}(Q_T)$

$$(2.10) \qquad q_\varepsilon = q_\varepsilon(x,t) = \int_x^{R_1} p_\varepsilon(\xi,t)\,d\xi \qquad \text{in } Q_T.$$

Then we have:

<u>Proposition 2.1</u> <u>Let $\varepsilon \to 0$</u>

$$(2.11) \qquad p_\varepsilon \to -\frac{\partial \pi^0}{\partial x}(\psi(t),t)\cdot\delta(x-\psi(t)) \qquad \text{in } \mathcal{D}'(Q_T),$$

$$(2.12) \qquad q_\varepsilon \to -\frac{\partial \pi^0}{\partial x}(\psi(t),t)\cdot(1-\chi(x,t)) \qquad \text{in } \mathcal{D}'(Q_T).$$

The proof for the elliptic case was given in [6]. If we repeat almost the same arguments as in [6], we can show (2.11)-(2.12).

The term q_ε is now called the integrated penalty. $q_\varepsilon(\psi(t),t)$ approximates the flux of $\pi^0(x,t)$ through the moving boundary $x = \psi(t)$.

2.2 <u>The first penalized problem</u>

H. kawarada and M. Natori [5] transformed $(Pr)_0$ into an initial boundary value problem defined in a cylindrical domain occupied jointly by water and ice for the heat equation with an artificial heat absorption in the ice region; we have the <u>Problem $(Pr)_1$</u>;

$$\text{To find: } \begin{cases} \theta^\varepsilon \in H^{2,1}(Q_T) \cap C^{1,0}(\bar{Q}_T)^* \\ s^\varepsilon \in C^1(0,T) \\ Q_T = (0,R_0)\times(0,T) \end{cases}$$

---

* $C^{m,n}(\bar{Q}_T)$, m, n *integer* is the space:

$$\{\phi \quad C(\bar{Q}_T),\ D_x^\mu\phi \in C(\bar{Q}_T)\ (0 \leq \mu \leq m),\ D_t^\gamma\phi \in C(\bar{Q}_T)\ (0 \leq \gamma \leq n)\}.$$

such that: for any $\varepsilon > 0$,

(2.13) $$\frac{\partial\theta^\varepsilon}{\partial t} = \frac{\partial^2\theta^\varepsilon}{\partial x^2} - \frac{1}{\varepsilon}\chi_\varepsilon\,\theta^\varepsilon \qquad \text{in } Q_T,$$

(2.14) $$\theta^\varepsilon(0,t) = f(t) \qquad 0 < t < T,$$

(2.15) $$\frac{\partial\theta^\varepsilon}{\partial x}(R_0,t) = 0 \qquad 0 < t < T,$$

(2.16) $$\theta^\varepsilon(x,0) = \tilde{\theta}_0(x) \qquad 0 < x < R_0,$$

(2.17) $$\frac{ds^\varepsilon}{dt} = \int_0^{R_0} \left(\frac{1}{\varepsilon}\chi_\varepsilon\,\theta^\varepsilon\right)(\xi,t)\,d\xi + h(t) \qquad 0 < t < T,$$

(2.18) $$s^\varepsilon(0) = b \quad ; \quad s^\varepsilon(t) < R_0 \quad \text{for any } t \in [0,T].$$

(2.19) $$\chi_\varepsilon = \chi_\varepsilon(x,t) = \begin{cases} 0 & \text{in } 0 < x \leqq s^\varepsilon(t), \quad 0 < t < T, \\ 1 & \text{in } s^\varepsilon(t) < x < R_0, \quad 0 < t < T. \end{cases}$$

The presence of the penalty term $-\frac{1}{\varepsilon}\chi_\varepsilon\theta^\varepsilon$ in (2.13) can approximately replace (1.3) if ε is sufficiently small. Further we should note that the integrated penalty of (2.17) approximates $-\frac{\partial\theta}{\partial x}(s(t),t)$ owing to proposition 2.1.

As a practical application of this method in the two dimensional case, we refer to a study of the behaviors of frozen soil in a neighbourhood of an underground storage tank of LNG (Liquid Natural Gas), which is kept at the temperature of -162°C. A program of this method was worked out and was established for practical uses by I. Yanagisawa [13].

2.3 The second penalized problem

Let us use the similar transformation as in (1.7).

We define:

$$y^{\varepsilon}(x,t) = \int_x^{R_0} \theta^{\varepsilon}(\xi,t)\,d\xi. \tag{2.20}$$

By integrating the equation (2.13) on $(0,R_0)$, we obtain:

$$\frac{\partial y^{\varepsilon}}{\partial t} = \frac{\partial^2 y^{\varepsilon}}{\partial x^2} - \frac{1}{\varepsilon}\int_x^{R_0} \chi_{\varepsilon}(\xi,t)\theta^{\varepsilon}(\xi,t)\,d\xi. \tag{2.21}$$

Now we have:

$$A^{\varepsilon} = \frac{1}{\varepsilon}\int_x^{R_0} \chi_{\varepsilon}\theta^{\varepsilon}\,d\xi = -\frac{1}{\varepsilon}\int_x^{R_0} \chi_{\varepsilon}\,\frac{\partial y^{\varepsilon}}{\partial \xi}(\xi,t)\,d\xi. \tag{2.22}$$

If $s^{\varepsilon}(t) \leq x \leq R_0$,

$$A^{\varepsilon} = -\frac{1}{\varepsilon}\int_x^{R_0} \frac{\partial y^{\varepsilon}}{\partial \xi}\,d\xi = \frac{1}{\varepsilon}y^{\varepsilon}(x,t). \tag{2.23}$$

If $0 < x < s^{\varepsilon}(t)$,

$$A^{\varepsilon} = -\frac{1}{\varepsilon}\int_{s^{\varepsilon}(t)}^{R_0} \frac{\partial y^{\varepsilon}}{\partial \xi}\,d\xi = \frac{1}{\varepsilon}y^{\varepsilon}(s^{\varepsilon}(t),t). \tag{2.24}$$

Further by (2.17), we have:

$$\frac{ds^{\varepsilon}}{dt} = -\frac{1}{\varepsilon}\int_0^{R_0} \chi_{\varepsilon}\,\frac{\partial y^{\varepsilon}}{\partial x}\,dx + h(t) \tag{2.25}$$

$$= \frac{1}{\varepsilon}\,y^{\varepsilon}(s^{\varepsilon}(t),t) + h(t).$$

Therefore, as y^{ε} is a decreasing function in x, we deduce:

$$(2.26) \quad A^\varepsilon = \frac{1}{\varepsilon} y^\varepsilon(s^\varepsilon(t),t)(1-\chi_\varepsilon) + \frac{1}{\varepsilon}\chi_\varepsilon y^\varepsilon(x,t)$$

$$= \frac{1}{\varepsilon} y^\varepsilon(s^\varepsilon(t),t) + \frac{1}{\varepsilon}\chi_\varepsilon(y^\varepsilon(x,t) - y^\varepsilon(s^\varepsilon(t),t))$$

$$= \frac{ds^\varepsilon}{dt} - h(t) - \frac{1}{\varepsilon}(y^\varepsilon - \varepsilon(\frac{ds^\varepsilon}{dt} - h(t)))^-.$$

Thus we have the following nonlinear problem: (Problem $(Pr)_2$)

To find $\begin{cases} y^\varepsilon \in C^{2,1}(Q_T), \\ s^\varepsilon \in C^2(0,T) \end{cases}$

such that:

$$(2.27) \quad \frac{\partial y^\varepsilon}{\partial t} = \frac{\partial^2 y^\varepsilon}{\partial x^2} + \frac{1}{\varepsilon}(y^\varepsilon - \varepsilon(\frac{ds^\varepsilon}{dt} - h))^- - \frac{ds^\varepsilon}{dt} + h \quad \text{in } Q_T,$$

$$(2.28) \quad \frac{\partial y^\varepsilon}{\partial x}(0,t) = -f(t) \qquad 0 < t < T,$$

$$(2.29) \quad y^\varepsilon(R_0,t) = 0 \qquad 0 < t < T,$$

$$(2.30) \quad y^\varepsilon(x,0) = y_0(x) \qquad 0 < x < R_0,$$

$$(2.31) \quad \frac{ds^\varepsilon}{dt} = \frac{1}{\varepsilon} y^\varepsilon(s^\varepsilon(t),t) + h(t) \qquad 0 < t < T,$$

$$(2.32) \quad s^\varepsilon(0) = b.$$

3. SOME RESULTS OF EXISTENCE, UNIQUENESS AND CONVERGENCE

Theorem 3.1 *Under the assumption (A1), $(Pr)_1$ has a unique solution $\{\theta^\varepsilon, s^\varepsilon(t)\}$ which satisfies:*

$$\begin{cases} \theta^{\varepsilon} \in H^{2,1}(Q_T) \cap C^{1,0}(\bar{Q}_T); \\ s^{\varepsilon}(t) \in C^1(0,T). \end{cases}$$

Theorem 3.2 Under the assumption (A1), $(Pr)_2$ has a unique solution $\{y^{\varepsilon}, s^{\varepsilon}(t)\}$ which satisfies:

$$\begin{cases} y^{\varepsilon} \geq 0, \qquad y^{\varepsilon} \in C^{2,1}(Q_T); \\ s^{\varepsilon} \in C^2(0,T). \end{cases}$$

Theorem 3.3 Under the assumption (A1), let $\varepsilon \to 0$, then the solution $\{y^{\varepsilon}; s^{\varepsilon}(t)\}$ of $(Pr)_2$ converges to a solution $\{y^0, s^0(t)\}$ of the (V.I) such that:

i) $y^{\varepsilon} \rightharpoonup y^0$ weakly in $L^2(0,T; H^2(\Omega))$,

ii) $s^{\varepsilon} \rightharpoonup s^0$ weakly in $H^1(0,T)$,

$s^{\varepsilon} \to s^0$ strongly in $C^{\tau}(0,T)$*, $(0 < \tau \leq \frac{1}{2})$.

Theorem 3.4 Under the assumption (A1), let $\varepsilon \to 0$, then the solution $\{\theta^{\varepsilon}; s^{\varepsilon}(t)\}$ converges to the unique solution of $(Pr)_0$ such that:

i) $\theta^{\varepsilon} \rightharpoonup \tilde{\theta}^0$ weakly in $H^1(Q_T)$,

$\theta^{\varepsilon}(\cdot,t) \to \tilde{\theta}^0(\cdot,t)$ strongly in $L^2(\Omega)$ for any $t \in [0,T]$;

ii) $s^{\varepsilon} \rightharpoonup s^0$ weakly in $H^1(0,T)$,

$s^{\varepsilon} \to s^0$ strongly in $C^{\tau}(0,T)$, $(0 \leq \tau \leq \frac{1}{2})$.

---

* $C^{\tau}(0,T) \equiv C^{0,\tau}(0,T)$

$\tilde{\theta}^0$ is the zero-extension of θ^0 into Q_T^1.

<u>Theorem 3.5</u> <u>Under the assumption (A.1)</u>, <u>the (V.I) has a unique solution</u> $y^0 \in C^{3,1}(Q_T)$ <u>and</u> $s^0(t) = \mathrm{Inf}\{x; y^0(x,t) = 0\} \in C^1(0,T)$.

By means of Theorem 3.3, we conclude that $(Pr)_2$ is interpreted as a penalized problem associated with the (V.I). Therefore we have a parallel structure, one level of which is a penalization with respect to $(Pr)_0$ and the other level is with respect to the (V.I).

Thus we have the general scheme:

Method of integrated penalty

$$\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{\begin{array}{l}(Pr)_0: \\ (1.1)-(1.6)\end{array}} \arrow[r, shift left] \arrow[d, "{y(x,t) = \int_x^{R_0} \tilde{\theta}(\xi,t)\,d\xi}"] & \boxed{\begin{array}{l}(Pr)_1: \\ (2.13)-(2.19)\end{array}} \arrow[l, shift left] \arrow[d, "{y^\varepsilon(x,t) = \int_x^{R_0} \theta^\varepsilon(\xi,t)\,d\xi}"] \\
\boxed{\begin{array}{l}V.I: \\ (1.9)-(1.12)\end{array}} & \boxed{\begin{array}{l}(Pr)_2: \\ (2.27)-(2.32)\end{array}} \arrow[l]
\end{tikzcd}$$

4. PROOFS OF THEOREMS

4.1 Proof of Theorem 3.1.

We put:

$$R_0 = b + \int_0^b \theta_0(x)\,dx + C_1(b,T) + \beta T,$$

$$C_\varepsilon = \frac{C_0 R_0}{\varepsilon} + \beta,$$

and

$$C_0 = \sup(\alpha, \gamma).$$

The constant $C_1(b,T)$ is defined in the appendix I.

Define a convex set K_1 in $C[0,T]$:

$$K_1 = \{\theta \in C[0,T]\ ;\ b < \phi(t_1) \leqq \phi(t_2) < R_0 \quad \text{for } 0 < t_1 \leqq t_2 < T \quad \text{and} \quad \phi(0) = b\}.$$

First let us study some regularity properties of a solution of the initial boundary value problem (2.13)-(2.16), in which the characteristic function χ_ε is replaced by:

$$\chi_s = \chi_s(x,t) = \begin{cases} 0 & \text{in } 0 < x \leqq s(t), \quad 0 < t < T, \\ 1 & \text{in } s(t) < x < R_0, \quad 0 < t < T, \end{cases}$$

$$\text{for some } s(t) \in K_1.$$

For simplicity, we denote this problem by $(Pr)_s$.

<u>Proposition 4.1</u> <u>$(Pr)_s$ has a solution θ which satisfies:</u>

(4.1) $\qquad \theta \in H^{2,1}(Q_T)$

(4.2) $\theta \in C^{1,0}(\bar{Q}_T)$.

Proof It is easy to prove the existence of the solution $\theta \in H^{2,1}(Q_T)$. Here we focus on the proof of (4.2). Let us introduce a function $\chi^\delta \in C^1(R^1)$ for some $\delta \in (0,\delta_0]$ $(\delta_0 > 0)$ such that:

i) $\frac{d\chi^\delta}{d\xi} \geqq 0$ for $\xi \in R^1$

ii) $$\chi^\delta(\xi) = \begin{cases} 0 & (\xi \leqq 0), \\ > 0 & (0 < \xi < \delta), \\ 1 & (\delta \leqq \xi). \end{cases}$$

Obviously, $\chi^\delta(x-s(t)) \to \chi_s$ in $L^2(Q_T)$ as $\delta \to 0$. Let θ^δ be the solution of the initial boundary value problem, in which χ_s of (2.13) in $(Pr)_s$ is replaced by $\chi^\delta(x-s(t))$. That is to say, $\theta^\delta \in C^{2,1}(Q_T)$ satisfies $\frac{\partial\theta^\delta}{\partial t} = \frac{\partial^2\theta^\delta}{\partial x^2} - \frac{1}{\varepsilon}\chi^\delta(x-s(t))\cdot\theta^\delta$ in Q_T. If we refer to Theorem 4.1 in §4 of chapter V in [7], we have:

(4.3) $$\left|\frac{\partial\theta^\delta}{\partial x}(x_1,t_1) - \frac{\partial\theta^\delta}{\partial x}(x_2,t_2)\right| \leqq C_5\{|x_1-x_2|^\alpha + |t_1-t_2|^{\alpha/2}\}$$

for any (x_1,t_1) and $(x_2,t_2) \in \bar{Q}_T$,

(4.4) $$\max_{(x,t)\in\bar{Q}_T}\left|\frac{\partial\theta^\delta}{\partial x}(x,t)\right| \leqq C_6.$$

Here we should note that C_5, C_6 and α do not depend on $\delta \in (0,\delta_0]$. By the theorem of Ascoli-Arzela, it follows that there exists a subsequence, still denoted θ^δ, such that:

$$(4.5)\qquad \theta^{\delta} \to \theta \qquad \text{in } C(\overline{Q}_T),$$

$$(4.6)\qquad \frac{\partial\theta^{\delta}}{\partial x} \to \frac{\partial\theta}{\partial x} \qquad \text{in } C(\overline{Q}_T), \qquad \text{when } \delta \to 0,$$

which implies (4.2). ∎

Proposition 4.2. The solution $\theta = \theta(x,t)$ of $(Pr)_s$ has the following estimates in Q_T:

$$(4.7)\qquad 0 \leq \theta \leq C_0 \qquad \text{in } \overline{Q}_T$$

$$(4.8)\qquad \int_0^{R_0} \left|\frac{\partial\theta}{\partial x}(\cdot,t)\right|^2 dx \leq C_2 \qquad \text{for any } t \in (0,T),$$

$$(4.9)\qquad \iint_{Q_T} \left|\frac{\partial\theta}{\partial t}\right|^2 dx\,dt \leq C_3,$$

$$(4.10)\qquad \iint_{Q_T} \left|\frac{\partial^2\theta}{\partial x^2}\right|^2 dx\,dt \leq C_4,$$

where the constants C_0, C_2 and C_3 are independent of ε, but C_4 depends on ε. All the constants C_0, C_2, C_3 and C_4 are independent of s ($\in K_1$).

Multiplying both sides of (2.13) by $\frac{\partial\theta}{\partial t}$ and integrating in Q_T, we easily obtain (4.8) and (4.9). Here we used (I.10) in the appendix I. By (4.8) and (4.9), we have (4.10). By using the maximum principle, we easily prove (4.7) ∎

Now we define a transformation A such that:

$$(4.11)\qquad s \in K_1 \to r = A\,s$$

$$(4.12)\qquad r = r(t) = b + \frac{1}{\varepsilon}\int_0^t \left[\int_0^{R_0} (\chi_s\theta)(\xi,\tau)\,d\xi\right]d\tau + \int_0^t h(\tau)\,d\tau,$$

where θ is the solution of $(Pr)_s$.

Let us verify some properties of A.

<u>Lemma 4.1.</u> <u>For any</u> $s \in K_1$,

(4.13) $$A s \in K_1.$$

<u>Proof</u> First, we easily prove that r(t) satisfies:

$$b < r(t_1) \leq r(t_2) \qquad \text{for } 0 < t_1 \leq t_2 < T$$

because $\theta \geq 0$ and $b > 0$.

Next, by an integration of the equation satisfied by θ^δ in Q_t, we obtain:

(4.14) $$\int_0^{R_0} \theta^\delta (x,t)\,dx + \frac{1}{\varepsilon}\int_0^t [\int_0^{R_0} (\chi^\delta \theta^\delta)(x,\tau)\,dx]\,d\tau$$
$$\leq \int_0^b \theta_0(x)\,dx + \operatorname*{Max}_{0\leq\tau\leq t} \left|\frac{\partial\theta^\delta}{\partial x}(\tau,0)\right| \qquad \text{in } 0 \leq t \leq T.$$

Note that the last term of the right side of (4.14) is estimated as follows:

(4.15) $$\operatorname*{Max}_{0\leq t\leq T} \left|\frac{\partial\theta^\delta}{\partial x}(t,0)\right| \leq C_1(b,T) \quad \text{for any } \delta \in (0,\delta_0].$$

The proof of (4.15) is given in the appendix I. Let $\delta \to 0$ in (4.14), we have:

(4.16) $$\frac{1}{\varepsilon}\int_0^t [\int_0^{R_0} (\chi_s \theta)(x,\tau)\,dx]\,d\tau \leq \int_0^b \theta_0(x)\,dx + C_1(b,T)$$
$$\text{for } 0 \leq t \leq T.$$

Then, by (4.12) and (4.16), we have (4.13). ∎

Lemma 4.2. A is continuous in K_1.

Proof Let θ (resp. θ') be solutions of $(Pr)_s$ (resp. $(Pr)_{s'}$) for $s, s' \in K_1$ and let $r = As$ and $r' = As'$. We have:

$$(4.17) \qquad |r - r'|_{C[0,T]} \leq \frac{C_0}{\varepsilon} \cdot T|s - s'|_{C[0,T]} + \frac{T}{\varepsilon} \cdot R_0 |\theta - \theta'|_{C(\bar{Q}_T)}.$$

On the other hand, we have:

$$(4.18) \qquad |\theta - \theta'|_{C(\bar{Q}_T)} \leq \frac{C_0\sqrt{2R_0T}}{\varepsilon}\sqrt{|s-s'|_{C[0,T]}}.$$

The proof of (4.18) is given in the appendix II. With (4.17) and (4.18), we easily obtain the continuity of A in K_1. ∎

Lemma 4.3. A is compact in K_1.

Proof From (4.12), we have:

$$(4.19) \qquad \frac{dr}{dt} = \frac{1}{\varepsilon}\int_0^{R_0} (\chi_s\theta)(s,t)\,dx + h(t) \qquad \text{in } 0 \leq t \leq T.$$

Then:

$$(4.20) \qquad 0 \leq \frac{dr}{dt} \leq C_\varepsilon \qquad \text{in } 0 \leq t \leq T,$$

Which implies the compactness of A in K_1. ∎

Proof of Theorem 3.1 K_1 is a bounded, closed convex in $C[0,T]$. By the Lemmas (4.1)-(4.3), we see that A is continuous compact from K_1 into itself. Then we can use Schauder's fixed point theorem. So there exists, at least, one fixed point $s \in K_1$ such that:

$$(4.21) \qquad s = As \qquad \text{in } K_1.$$

Thus $(Pr)_1$ has, at least, one solution which satisfies $\theta \in H^{2,1}(Q_T)$, $\theta \in C^{1,0}(\bar{Q}_T)$ and $x = s(t) \in C^1(0,T)$. The uniqueness of the solution of $(Pr)_1$ will be proved at the end of the proof of Theorem 3.2. ∎

4.2 <u>Proof of Theorem 3.2.</u>

Let $\{\theta(x,t), s(t)\}$ be the solution of $(Pr)_1$. If we refer to the paragraph 2.3, then:

$$(4.22) \qquad y = y(x,t) = \int_x^{R_0} \theta(\xi,t)\,d\xi \geq 0$$

and $x = s(t)$ satisfy $(Pr)_2$. Because of $y \in C^{2,0}(Q_T)$ and $s \in C^1[0,T]$, we see that:

$$(4.23) \qquad \frac{\partial y}{\partial t} \in C(\bar{Q}_T).$$

Furthermore, we have:

$$(4.24) \qquad \frac{d^2 s}{dt^2} = \frac{1}{\varepsilon}\frac{\partial y}{\partial x}(s(t),t)\frac{ds}{dt} + \frac{1}{\varepsilon}\frac{\partial y}{\partial t}(s(t),t) + \frac{dh}{dt} \qquad \text{in } 0 \leq t \leq T$$

which implies:

$$(4.25) \qquad s \in C^2(0,T).$$

Therefore $(Pr)_2$ has, at least, one classical solution $\{y \in C^{2,1}(Q_T);\ x = s(t) \in C^2(0,T)\}$.

Suppose that $(Pr)_2$ has two solutions $\{y_i; s_i(t)\}$ for $i = 1,2$. Let us estimate the difference between $\frac{ds_1}{dt}$ and $\frac{ds_2}{dt}$; we have:

$$(4.26)\qquad \left|\frac{ds_1}{dt} - \frac{ds_2}{dt}\right| \le \frac{1}{\varepsilon}|y_1(s_1(t),t) - y_2(s_1(t),t)|$$

$$+ \frac{1}{\varepsilon}|y_2(s_1(t),t) - y_2(s_2(t),t)|$$

$$\le \frac{1}{\varepsilon}|y_1 - y_2|_{C(\bar{Q}_T)}$$

$$+ \frac{1}{\varepsilon}\left|\frac{\partial y_2}{\partial x}\right|_{C(\bar{Q}_T)} \cdot |s_1 - s_2|_{C[0,T]}.$$

Because

$$(4.27)\qquad \left|\frac{\partial y_2}{\partial x}\right|_{C(\bar{Q}_T)} \le C_0,$$

we have

$$(4.28)\qquad \left|\frac{ds_1}{dt} - \frac{ds_2}{dt}\right|_{C[0,T]} \le \frac{1}{\varepsilon}|y_1 - y_2|_{C(\bar{Q}_T)}$$

$$+ \frac{C_0}{\varepsilon}|s_1 - s_2|_{C[0,T]}.$$

We have the following inequality (see the appendix III):

$$(4.29)\qquad |y_1 - y_2|_{C(\bar{Q}_T)} \le \int_0^T \{\frac{1}{\varepsilon}|y_1 - y_2|_{C(\bar{Q}_T)}$$

$$+ 2\left|\frac{ds_1}{dt} - \frac{ds_2}{dt}\right|_{C[0,T]}\}dt.$$

From (4.29), we obtain:

$$(4.30)\qquad |y_1 - y_2|_{C(\bar{Q}_{T_0})} \le \frac{2T_0\varepsilon}{\varepsilon - T_0}\cdot\left|\frac{ds_1}{dt} - \frac{ds_2}{dt}\right|_{C[0,T_0]}$$

$$\text{for } 0 < T_0 < \varepsilon.$$

Using (4.28) together with (4.30), we have

$$(4.31) \qquad \left|\frac{ds_1}{dt} - \frac{ds_2}{dt}\right|_{C[0,T_0]}$$

$$\leq \left(\frac{2T_0}{\varepsilon - T_0} + \frac{C_0 T_0}{\varepsilon}\right)\left|\frac{ds_1}{dt} - \frac{ds_2}{dt}\right|_{C[0,T_0]}$$

for $0 < T_0 < \varepsilon$.

Choose $T_0 = T_0(\varepsilon)$ to satisfy:

$$(4.32) \qquad 0 < \frac{2T_0}{\varepsilon - T_0} + \frac{C_0 T_0}{\varepsilon} < 1 \qquad \text{for any } \varepsilon > 0.$$

That is:

$$(4.33) \qquad 0 < T_0(\varepsilon) < \mathrm{Inf}\{\frac{\varepsilon}{2C_0}(3 + C_0 - \sqrt{C_0^2 + 2C_0 + 9}), \varepsilon\}.$$

Then we have a contradiction and we deduce

$$(4.34) \qquad \frac{ds_1}{dt}(t) = \frac{ds_2}{dt}(t) \qquad \text{in } 0 \leq t \leq T_0(\varepsilon).$$

Because $s_1(0) = s_2(0) = b$, we have

$$(4.35) \qquad s_1(t) = s_2(t) \qquad \text{in } 0 \leq t \leq T_0(\varepsilon).$$

With (4.33) and the same procedure $\{[\frac{T}{T_0(\varepsilon)}] + 1\}$ times,
We obtain:

$$(4.36) \qquad s_1(t) = s_2(t) \qquad \text{in } 0 \leq t \leq T,$$

which implies:

$$(4.37) \qquad y_1(x,t) = y_2(x,t) \qquad \text{in } \overline{Q}_T.$$

Thus this proves the uniqueness of the solution of $(Pr)_2$. From this uniqueness, we easily obtain the uniqueness of the solution of $(Pr)_1$. Let $\{\theta_1, s_1(t)\}$ and $\{\theta_2, s_2(t)\}$ be two solutions of $(Pr)_1$. Then

$$y_1 = \int_x^{R_0} \theta_1(\xi,t)\,d\xi \quad \text{and} \quad y_2 = \int_x^{R_0} \theta_2(\xi,t)\,d\xi$$

satisfy $(Pr)_2$. From the uniqueness of the solution of $(Pr)_2$, we deduce:

$$(4.38) \qquad s_1(t) = s_2(t) \qquad \text{in } 0 \le t \le T.$$

On the other hand, we have,

$$(4.39) \qquad \int_x^{R_0} (\theta_1(\xi,t) - \theta_2(\xi,t)\,d\xi = 0$$

for any $(x,t) \in \overline{Q}_T$.

If we differentiate (4.39) with respect to x, we have

$$(4.40) \qquad \theta_1(x,t) = \theta_2(x,t) \qquad \text{in } \overline{Q}_T.$$

4.3 Proof of Theorem

Here we denote by $\{y^\varepsilon ; s^\varepsilon(t)\}$ the solution of $(Pr)_2$.

Lemma 4.4. y^ε verifies the following estimates:

$$(4.41) \qquad -C_0 \le \frac{\partial y^\varepsilon}{\partial x} \le 0 \quad \text{and} \quad 0 \le y^\varepsilon \le C_0R_0 \quad \text{in } \overline{Q}_T,$$

$$(4.42) \qquad \int_0^{R_0} \left| \frac{\partial^2 y^\varepsilon}{\partial x^2}(\cdot,t) \right|^2 dx \le C_2 \quad \text{for any } t \in (0,T),$$

$$(4.43) \qquad \iint_{Q_T} \left| \frac{\partial y^\varepsilon}{\partial t} \right|^2 dx\,dt \le C_3,$$

$$(4.44) \qquad \iint_{Q_T} \left|\frac{\partial^3 y}{\partial x^3}\right|^2 dx\,dt \leq C_4.$$

We easily obtain (4.41)-(4.44) from (4.7)-(4.10) in Proposition 4.2.

Lemma 4.5. $s^\varepsilon(t)$ verifies the estimate:

$$(4.45) \qquad \left|\frac{ds^\varepsilon}{dt}\right|_{L^2(0,T)} \leq C_7,$$

where C_7 is independent of ε.

Proof From (2.27), we have:

$$(4.46) \qquad \frac{ds^\varepsilon}{dt} = h - \frac{\partial y^\varepsilon}{\partial t} + \frac{\partial^2 y^\varepsilon}{\partial x^2} \qquad \text{in } 0 < x < s^\varepsilon(t),$$

$$0 < t < T.$$

From (4.46), we have

$$(4.47) \qquad b\int_0^T \left|\frac{ds^\varepsilon}{dt}\right|^2 dt \leq 3\Big[\iint_{Q_T}\left|\frac{\partial y^\varepsilon}{\partial t}\right|^2 dx\,dt + \iint_{Q_T}\left|\frac{\partial^2 y^\varepsilon}{\partial x^2}\right|^2 dx\,dt + R_0\cdot T\cdot\beta^2\Big].$$

By (4.42), (4.43) and (4.47), we deduce:

$$(4.48) \qquad \left|\frac{ds^\varepsilon}{dt}\right|_{L^2(0,T)} \leq \sqrt{\frac{3}{b}}\sqrt{C_2 T + C_3 + R_0 T\beta^2} = C_7. \quad \blacksquare$$

Lemma 4.6. If $\varepsilon \to 0$, then $\{y^\varepsilon, x = s^\varepsilon(t)\}$ converges to $\{y^0, s^0(t)\}$ in the following topology:

(4.49) $s^\varepsilon \to s^0$ strongly in $L^2(0,T)$,

strongly in $C^\tau(0,T)$, $(0 \leqq \tau \leqq \frac{1}{2})$

(4.50) $\frac{ds^\varepsilon}{dt} \rightharpoonup \frac{ds^0}{dt}$ weakly in $L^2(0,T)$,

(4.51) $y^\varepsilon \to y^0 (\geqq 0)$ strongly in $L^2(Q_T)$,

(4.52) $\frac{\partial y^\varepsilon}{\partial x} \to \frac{\partial y^0}{\partial x}$ strongly in $L^2(Q_T)$,

(4.53) $\frac{\partial^2 y^\varepsilon}{\partial x^2} \rightharpoonup \frac{\partial^2 y^0}{\partial x^2}$ weakly in $L^2(0,T; L^2(0,R_0))$,

(4.54) $\frac{\partial y^\varepsilon}{\partial t} \rightharpoonup \frac{\partial y^0}{\partial t}$ weakly in $L^2(0,T; L^2(0,R_0))$.

Define a convex set K_2 such that

$$K_2 = \{\phi \in L^2(\Omega) ; \phi \geqq 0 \quad \text{a.e. in } \Omega\}.$$

If we multiply (2.27) by $(\phi - y^\varepsilon)$, $\phi \in K_2$, and integrate in Q_T, then we have:

$$(4.55) \quad \int_0^T (\frac{\partial y^\varepsilon}{\partial t}, \phi - y^\varepsilon) dt - \int_0^T (\frac{\partial^2 y^\varepsilon}{\partial x^2}, \phi - y^\varepsilon) dt$$

$$- \frac{1}{\varepsilon}\int_0^T ((y^\varepsilon + \varepsilon a_\varepsilon)^-, \phi - y^\varepsilon) dt = \int_0^T (a_\varepsilon, \phi - y^\varepsilon) dt$$

$$\forall \phi \in K_2.$$

Where $a_\varepsilon = - \frac{ds^\varepsilon}{dt} + h(t)$.

Now, using the monotone property of $(y^\varepsilon + \varepsilon a_\varepsilon)^-$, and because $\phi \in K_2$, we have:

(4.56) $$-\frac{1}{\varepsilon}((y^{\varepsilon}+\varepsilon a_{\varepsilon})^{-},\phi - y^{\varepsilon})$$

$$= \frac{1}{\varepsilon}(\phi^{-} - (y^{\varepsilon}+\varepsilon a_{\varepsilon})^{-},\phi - (y_{\varepsilon}+\varepsilon a_{\varepsilon}))$$

$$- ((y^{\varepsilon}+\varepsilon a_{\varepsilon})^{-},a_{\varepsilon})$$

$$\leq -((y^{\varepsilon}+\varepsilon a_{\varepsilon})^{-},a_{\varepsilon}).$$

Using (4.45), we have

(4.57) $$\varepsilon a_{\varepsilon} \to 0 \quad \text{strongly in } L^2(0,T) \quad \text{as} \quad \varepsilon \to 0 .$$

By (4.51) and (4.56), we have

(4.58) $$(y^{\varepsilon}+\varepsilon a_{\varepsilon})^{-} \to (y^{0})^{-} = 0 \quad \text{strongly in } L^2(Q_T) \quad \text{as} \quad \varepsilon \to 0 .$$

From (4.58), we have

(4.59) $$\int_0^T((y^{\varepsilon}+\varepsilon a_{\varepsilon})^{-},a_{\varepsilon})dt \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

Therefore, using Lemma 4.6, (4.56) and (4.59), we obtain at the limit in (4.55):

(4.60) $$\int_0^T(\frac{\partial y^0}{\partial t}, \phi-y^0)dt - \int_0^T(\frac{\partial^2 y^0}{\partial x^2}, \phi-y^0)dt$$

$$\geq \int_0^T(a_0,\phi-y^0)dt.$$

Here $a_0 = -\frac{ds^0}{dt} + h$. The partial integration of the second term of the left side of (4.60) yields (1.9).

4.4 Proof of Theorem 3.4.

For simplicity, we assume $f \equiv 0$ in $(0,T)$.

Step 1; convergence of θ^{ε} and s^{ε}.

We already proved the convergence of $s^{\varepsilon}(t)$ in Lemma 4.6. From Proposition 4.2, we can extract a subsequence, still denoted θ^{ε}, such that:

(4.61) $$\theta^{\varepsilon} \rightharpoonup w \quad \text{weakly in } H^1(Q_T),$$

$$\theta^{\varepsilon} \to w \quad \text{strongly in } L^1(Q_T),$$

and we have ((4.8) Proposition 4.2):

(4.62) $$\int_0^{R_0} \left|\frac{\partial w}{\partial x}(\cdot,t)\right|^2 dx \le C_2 \qquad \forall t \in (0,T).$$

Define:

$$\Omega^0(t) = \{x\,;\, 0 < x < s^0(t)\} \qquad (0 < t < T),$$

$$\Omega^1(t) = \Omega - \overline{\Omega^0(t)}\ ,$$

$$Q_T^0 = \bigcup_{0<t<T} \Omega^0(t),$$

and $$Q_T^1 = \bigcup_{0<t<T} \Omega^1(t).$$

Lemma 4.7. The function w satisfies:

(4.63) $$w\big|_{Q_T^0} \in L^2(0,T\,;\,H_0^1(\Omega^0(t)));$$

(4.64) $$w\big|_{Q_T^1} = 0.$$

Proof First we prove (4.64). We have.

(4.65) $$\iint_{Q_T} \chi_\varepsilon |\theta^\varepsilon|^2 \, dx\, dt \leq \varepsilon \cdot C_8,$$

where C_8 is independent of ε. From (4.59) and (4.61), we get

(4.66) $$\iint_{Q_T^1} |w|^2 \, dx\, dt \equiv 0.$$

By using (4.62), we have

(4.67) $$|w(x_1,t) - w(x_2,t)| \leq \int_{x_1}^{x_2} \left|\frac{\partial w}{\partial x}(\cdot,t)\right| dx$$

$$\leq \sqrt{C_2}\,|x_1-x_2|^{\frac{1}{2}} \quad \forall x_1, x_2 \in (0,R_0), \quad \forall t \in (0,T),$$

which implies, with (4.64)

(4.68) $$w\big|_{x=s^0(t)} = 0 \qquad \forall t \in (0,T).$$

Since $w \in H^1(Q_T)$, we have (4.63). ∎

<u>Lemma 4.8.</u> $w\big|_{Q_T^0}$ <u>satisfies</u>:

(4.69) $$\iint_{Q_T^0} \frac{\partial w}{\partial t} \phi \, dx\, dt + \iint_{Q_T^0} \frac{\partial w}{\partial x} \frac{\partial \phi}{\partial x} \, dx\, dt = 0$$

$$\forall \phi \in L^2(0,T\,;\, H_0^1(\Omega^0(t))).$$

<u>Proof</u> For any ϕ, we can define ϕ^ε such that:

i) $\phi^\varepsilon \in H^{1,0}(Q_T)$; ii) $\phi^\varepsilon(0,t) = 0$ $(t \in (0,T))$;

iii) $\chi_\varepsilon \phi^\varepsilon = 0$ in Q_T ;

iv) $\phi^\varepsilon \to \tilde{\phi}$ strongly in $L^2(Q_T)$ and

$\frac{\partial \phi^\varepsilon}{\partial x} \to \frac{\partial \tilde{\phi}}{\partial x}$ strongly in $L^2(Q_T)$ when $\varepsilon \to 0$.

Here $\tilde{\phi}$ denotes the zero-extension of ϕ into Q_T^1. If we multiply (2.13) by ϕ^ε and if we integrate in Q_T, we have:

$$\iint_{Q_T} \frac{\partial \theta^\varepsilon}{\partial t} \phi^\varepsilon \, dx\, dt + \iint_{Q_T} \frac{\partial \theta^\varepsilon}{\partial x} \frac{\partial \phi^\varepsilon}{\partial x} dx\, dt = 0. \tag{4.70}$$

At the limit, we obtain,

$$\iint_{Q_T} \frac{\partial w}{\partial t} \tilde{\phi} \, dx\, dt + \iint_{Q_T} \frac{\partial w}{\partial x} \frac{\partial \tilde{\phi}}{\partial x} dx\, dt = 0. \tag{4.71}$$ ∎

Now let us consider the following initial boundary value problem $(Pr)_{s^0}$;

$$\begin{cases} \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} & \text{in } Q_T^0, \\ \theta(0,t) = 0 & 0 < t < T, \\ \theta(s^0(t),t) = 0 & 0 < t < T, \\ \theta(x,0) = \theta_0(x) & 0 < x < b. \end{cases}$$

Here we should note that $s^0 \in C^{\frac{1}{2}}(0,T)$. Refering to Friedman [2], we easily obtain that $(Pr)_{s^0}$ has the unique solution $\theta^0 \in C^{2,1}(Q_T^0)$ which satisfies $\frac{\partial \theta^0}{\partial x}(s^0(t),t) \in C(0,T)$.

<u>Lemma 4.9.</u> <u>w satisfies:</u>

$$w(x,t) = \theta^0(x,t) \quad \text{a.e. in } Q_T^0. \tag{4.72}$$

Proof Put $W = w - \theta$. Then W verifies:

$$(4.73) \qquad \iint_{Q_T^0} \frac{\partial W}{\partial t} \phi \, dx\, dt + \iint_{Q_T^0} \frac{\partial W}{\partial x} \frac{\partial \phi}{\partial x} \, dxdt = 0$$

$$\forall \phi \in L^2(0,T\,;\, H_0^1(\Omega^0(t))).$$

Put $\phi = W$ in (4.73). Then

$$(4.74) \qquad \frac{1}{2}|W(\cdot,T)|^2 + \iint_{Q_T^0} \left|\frac{\partial W}{\partial x}\right|^2 dx\, dt = 0,$$

which implies (4.72). ∎

We modify w on a null set so that (4.72) holds for every (x,t) in Q_T^0, i.e., $w(x,t) \equiv \theta^0(x,t)$.

Step 2; $\theta^0(x,t)$ and $s^0(t)$ satisfy the Stefan condition (1.5). Define

$$(4.75) \qquad q^\varepsilon = q^\varepsilon(x,t) = \int_x^{R_0} (\tfrac{1}{\varepsilon} \chi_\varepsilon \, \theta^\varepsilon)(\xi,t) d\xi.$$

Lemma 4.10. If $\varepsilon \to 0$, we have

$$(4.76) \qquad q^\varepsilon \rightharpoonup -\theta_x^0(s^0(t),t)(1 - \chi^0(x,t))$$

weakly in $L^2(Q_T)$.

Proof From (2.12) (Proposition 2.1), we obtain

$$(4.77) \qquad q^\varepsilon \to -\theta_x^0(s^0(t),t)(1 - \chi^0(x,t) \qquad \text{in } \mathcal{D}'(Q_T).$$

From (4.7) (Proposition 4.1), it follows that:

$$(4.78) \qquad \frac{ds^\varepsilon}{dt} - h = q^\varepsilon(0,t) \geq q^\varepsilon(x,t) \geq 0 \qquad \text{in } Q_T,$$

then we have:

$$(4.79) \qquad |q^{\varepsilon}|_{L^2(Q_T)} \leq \sqrt{R_0}\cdot(C_7+\beta\sqrt{T}).$$

With (4.79), (4.77) implies (4.76). ∎

Finally we show that θ^0 and $s^0(t)$ satisfy the Stefan condition (1.5). From (2.17), we have:

$$(4.80) \qquad \frac{ds^{\varepsilon}}{dt}(1-\chi_{\varepsilon}) = q^{\varepsilon}(1-\chi_{\varepsilon}) + h(1-\chi_{\varepsilon}) \quad \text{in } Q_T.$$

Integrating (4.80) in Q_t $(0 < t \leq T)$ and passing to the limit, we have:

$$(4.81) \qquad \iint_{Q_t^0} \frac{ds^0}{dt}\,dxd\tau = -\iint_{Q_t^0} \frac{\partial\theta^0}{\partial x}(s^0(\tau),\tau)\,dxd\tau + \iint_{Q_t^0} h\,dxd\tau.$$

Here we used (4.76). By Nikodym's theorem, we have

$$(4.82) \qquad s^0(t) = b - \int_0^t \frac{\partial\theta^0}{\partial x}(s^0(\tau),\tau)\,d\tau + \int_0^t h(\tau)\,d\tau.$$

Since $\frac{\partial\theta^0}{\partial x}(s^0(t),t) \in C[0,T]$, (4.82) implies (1.5). Because of the unique existence of the solution of $(Pr)_0$, all the sequence $\{\theta^{\varepsilon}(x,t), s^{\varepsilon}(t)\}$ converges to $\{\theta^0(x,t), s^0(t)\}$.

4.5 Proof of Theorem 3.5.

From Theorems 3.3 and 3.4, we have:

$$\theta^{\varepsilon} \to \theta^0 \qquad \text{strongly in } L^2(Q_T)$$

and $$\frac{\partial y^{\varepsilon}}{\partial x} \to \frac{\partial y^0}{\partial x} \qquad \text{strongly in } L^2(Q_T).$$

Since

$$(4.83) \qquad \frac{\partial y^\varepsilon}{\partial x} = -\theta^\varepsilon \qquad \text{a.e in } Q_T,$$

we have

$$(4.84) \qquad \tilde{\theta}^0 = -\frac{\partial y^0}{\partial x} \qquad \text{a.e in } Q_T.$$

We obtain

$$(4.85) \qquad 0 \equiv y^\varepsilon(R_0,\cdot) = y^0(R_0,\cdot) \in H^{\frac{1}{2}}(0,T).$$

From (4.84) and (4.85), we have

$$(4.86) \qquad y^0(x,t) = \int_x^{R_0} \tilde{\theta}^0(\xi,t)\,d\xi.$$

Since θ^0 verifies $0 < \theta^0 < C_0$ in Q_T^0, we have

$$(4.87) \qquad s^0(t) = \operatorname{Inf}\{x \,;\, y^0(x,t) = 0\} \qquad (0 < t < T).$$

Here $x = s^0(t)$ is the free boundary of $(Pr)_0$ corresponding to θ^0. $\{\theta^0, s^0(t)\}$ is the unique solution of $(Pr)_0$. Therefore the (V.I) has a unique solution.

5. NUMERICAL RESULTS

Here we present the numerical methods used to solve Problem $(Pr)_2$ and the (V.I). The results concerning $(Pr)_1$ were presented in H. Kawarada-M. Natori [5].

We use the following algorithms.

5.1 Problem $(Pr)_2$

1) Initialization.

Put $y_0^\varepsilon = y_0(x)$, $\quad s_0^\varepsilon = b \quad$ for $n = 0$.

2) $i = 0, \quad s_{n+1}^{\varepsilon,0} = s_n^{\varepsilon}$

3) Determine $y_{n+1}^{\varepsilon,1}$ solution

$$(5.1) \qquad \frac{y_{n+1,i}^{\varepsilon} - y_n^{\varepsilon}}{\Delta t} = \frac{d^2 y_{n+1}^{\varepsilon,i}}{dx^2} + \frac{1}{\varepsilon}\left(y_{n+1}^{\varepsilon,i} - \left(\frac{s_{n+1}^{\varepsilon,i} - s_n^{\varepsilon}}{\Delta t} - h^{n+1}\right)\right)^- - \frac{s_{n+1}^{\varepsilon,i} - s_n^{\varepsilon}}{\Delta t} + h^{n+1},$$

$$(5.2) \qquad \frac{dy_{n+1}^{\varepsilon,i}}{dx}(0) = -f^{n+1},$$

$$(5.3) \qquad y_{n+1}^{\varepsilon,i}(R_0) = 0.$$

4) Determine $\tilde{s}_{n+1}^{\varepsilon,i}$ by the expression:

$$\frac{\tilde{s}_{n+1}^{\varepsilon,i+1} - s_n^{\varepsilon}}{t} = \frac{1}{\varepsilon}\, y_{n+1}^{\varepsilon}(s_{n+1}^{\varepsilon,i}) + h^{n+1}$$

5) $s_{n+1}^{\varepsilon,i+1} = (1-\omega)\, s_{n+1}^{\varepsilon,i} + \omega\, \tilde{s}_{n+1}^{\varepsilon,i+1} \qquad (0 < \omega < 1)$

6) Test of convergence;

If the test is verified, $s_{n+1}^{\varepsilon} = s_{n+1}^{\varepsilon,i+1}$, $y_{n+1}^{\varepsilon} = y_{n+1}^{\varepsilon,i}$

If not, $i = i + 1$ go to 3.

7) If $n + 1 = N$ (number of step in t), stop.

If not, $n = n + 1$ go to 2.

The nonlinear equation (5.1)-(5.3) is approximated by a finite difference method (or a finite element method). This problem is solved by an over-relaxation method.

5.2 The Variational Inequality

1) Initialization $\quad y^0(x) = y_0(x), \quad s^0 = b \quad$ for $n = 0$.

2) $i = 0, \quad s_0^{n+1} = s^n$

3) Determine y_i^{n+1} solution of:

$$(5.4)\begin{cases} \left(\dfrac{y_i^{n+1} - y^n}{\Delta t}, \phi - y_i^{n+1}\right) + a(y_i^{n+1}, \phi - y_i^{n+1}) \\ \geq \left(-\dfrac{s_1^{n+1} - s^n}{\Delta t} + h^{n+1}, \phi - y_i^{n+1}\right) \\ \quad + f^{n+1}(\phi(0) - y_i^{n+1}(0)) \qquad \text{for any } \phi \in K_2. \\ y_i^{n+1} \in K_2 \end{cases}$$

4) $\tilde{s}_i^{n+1} = \mathrm{Inf}\{x \in (0,R_0) \,|\, y_i^{n+1}(x) = 0\}$

5) 6) 7) are the same operations as $(Pr)_2$. The variational inequality (5.4) is approximated by a finite difference method (or a finite element method). This problem is solved by a relaxation-projection method (R. Glowinski, J.L. Lions and R. Tremolieres [1]).

We tested these methods for the following problem, in one dimensional case:

$$\begin{cases} \dfrac{\partial\theta}{\partial t} = \dfrac{\partial^2\theta}{\partial x^2}, \\ \theta(0,t) = t + \frac{1}{2} \\ \theta(s(t),t) = 0 \\ \dfrac{\partial\theta}{\partial x}(s(t),t = -4\dfrac{ds}{dt} + \dfrac{2t}{\sqrt{3-2t}} \\ \theta(x,0) = \left(\dfrac{x^2}{2} - 2x + \frac{1}{2}\right)^+. \end{cases}$$

The exact solution is:

$$\theta(x,t) = \left(\frac{x^2}{2} - 2x + t + \frac{1}{2}\right)^+; \quad s(t) = 2 - \sqrt{3-2t} \ .$$

To solve $(Pr)_2$, we took $R_0 = T = 1$. We divided $(0,R_0)$ into 30 steps and $(0,T)$ into 20 steps. The results, presented here, were obtained for $\varepsilon = 10^{-3}$ and $\omega = 0.15$. The computer time was approximately equal to 4s on IBM 370-168. For the V.I, we took 30 steps of space and 10 steps of time. The coefficient ω was chosen to be equal to 0.3. The computer time was the same order as for the preceding method. The results on the free boundary were very comparable, which are shown in the following table:

t	Exact sol.	$(Pr)_2$	(V.I)
0	0.2679	0.2679	0.2668
0.1	0.3267	0.3303	0.3235
0.2	0.3875	0.3956	0.3845
0.3	0.4508	0.4576	0.4468
0.4	0.5168	0.5290	0.5135
0.5	0.5858	0.5977	0.5832
0.6	0.6584	0.6711	0.6578
0.7	0.7351	0.7488	0.7363
0.8	0.8168	0.8325	0.8196
0.9	0.9046	0.9208	0.9112
1	1.000	1.023	0.9965

We conclude a very good agreement of the numerical results with the exact solution.

On the other hand, the method of fixed point will be used. Particulary we notice the utilization of the method presented in M. Sermange [12].

6. *CONCLUSION*

The methods, presented here, have a fairly general character for the free boundary problems appearing in the system including change of phase. They permit, in particular, to take into account heat source along the free surface and give efficient numerical algorithms.

APPENDIX I

1st step We shall consider the following initial boundary value problems (A), (B) and (C):

$$(A)\quad \begin{cases} \dfrac{\partial\theta}{\partial t} = \dfrac{\partial^2\theta}{\partial x^2} & \text{in } 0<x<b,\ \ 0<t<T, \\ \theta(0,t) = f(t) & 0<t<T, \\ \theta(b,t) = 0 & 0<t<T, \\ \theta(x,0) = \theta_0(x) & 0<x<b, \end{cases}$$

$$(B)\quad \begin{cases} \dfrac{\partial\theta}{\partial t} = \dfrac{\partial^2\theta}{\partial x^2} - \dfrac{1}{\varepsilon}\chi^\delta\,(x-s(t))\theta & \text{in } 0<x<R, \\ & 0<t<T, \\ \theta(0,t) = f(t) & 0<t<T, \\ \theta(R,t) = 0 & 0<t<T, \\ \theta(x,0) = \tilde{\theta}_0(x) & 0<x<R, \end{cases}$$

$$(C)\quad \begin{cases} \dfrac{\partial\theta}{\partial t} = \dfrac{\partial^2\theta}{\partial x^2} & \text{in } 0<x<R,\ \ 0<t<T, \\ \theta(0,t) = f(t) & 0<t<T, \\ \theta_x(R,t) = 0 & 0<t<T, \\ \theta(x,0) = \tilde{\theta}_0(x) & 0<x<R. \end{cases}$$

Here R in (B) and (C) is an arbitrary number which satisfies $R>b>0$. s(t) in χ^δ of (B) satisfies

i) $s \in C^0[0,T]$;

ii) s(t) is monotone increasing in t ;

iii) $s(0) = b$, $S(T) < R$.

Let θ_1, θ^δ and θ_2 be the solutions respectively of (A), (B) and (C) under the assumption (A1). Then, from the maximum principle, we have:

$$\text{(I.1)} \qquad \theta_1(x,t) \leq \theta^\delta(x,t) \leq \theta_2(x,t) \qquad \text{in } 0 \leq x \leq b, \; 0 \leq t \leq T.$$

From (I.1) it follows:

$$\text{(I.2)} \qquad \left|\frac{\partial\theta^\delta}{\partial x}(0,t)\right|_{C^0[0,T]} \leq \operatorname{Sup}\left\{\left|\frac{\partial\theta_1}{\partial x}(0,t)\right|_{C^0[0,T]}, \left|\frac{\partial\theta_2}{\partial x}(0,t)\right|_{C^0[0,T]}\right\}.$$

<u>2nd step</u> We shall prove that $\left|\frac{\partial\theta_2}{\partial x}(0,t)\right|_{C^0[0,T]}$ is uniformly bounded for any R (>b).

We introduce Green's function which satisfies $\frac{\partial\theta}{\partial x}(R,t)=0$ $(0 < t < T)$:

$$\begin{aligned} N(x,t\,;\,\xi-R,\tau) &= \frac{1}{2\sqrt{\pi(t-\tau)}}\exp\left\{-\frac{(x-\xi+R)^2}{4(t-\tau)}\right\} \\ &+ \frac{1}{2\sqrt{\pi(t-\tau)}}\exp\left\{-\frac{(x+\xi-R)^2}{4(t-\tau)}\right\} \\ &= K(x,t\,;\,\xi-R,\tau) + K(x,t\,;\,-\xi+R,\tau). \end{aligned}$$

Integrating Green's identity:

$$\frac{\partial}{\partial\xi}\left(N\frac{\partial\theta_2}{\partial\xi} - \theta_2\frac{\partial N}{\partial\xi}\right) - \frac{\partial}{\partial\tau}(N\theta_2) = 0$$

on the domain $0 < \xi < R$, $0 < \xi < \tau < t-\varepsilon$ and letting $\varepsilon \to 0$, we obtain:

(I.3) $$\theta_2(R-x,t) = \int_0^b N(x,t\,;\,\xi-R,0)\theta_0(\xi)d\xi - \int_0^t N(x,t\,;\,-R,\tau)\frac{\partial\theta_2}{\partial\xi}(0,\tau)d\tau + \int_0^t \frac{\partial N}{\partial\xi}(x,t\,;\,-R,2)f(\tau)d\tau.$$

Let denote $v(\tau) = \frac{\partial\theta_2}{\partial\xi}(0,\tau)$. We differentiate (I.3) with respect to x and $x \to R-0$. We obtain:

(I.4) $$v(t) = 2\int_0^t v(\tau)\frac{\partial N}{\partial x}(R,t\,;\,-R,\tau)d\tau + 2\int_0^t G(R,t\,;\,-R,\tau)\dot{f}(\tau)d\tau - 2\int_0^b G(x,t\,;\,\xi-R,0)\theta_0(\xi)d\xi,$$

where $G(x,t;\xi,\tau)=K(x,t;\xi,\tau)-K(x,t;-\xi,\tau)$ and $\dot{f}(\tau)=\frac{df(\tau)}{d\tau}$.

We note that the following inequality holds:

(I.5) $$\int_0^t \left|\frac{\partial N}{\partial x}(R,t\,;\,-R,\tau)\right|d\tau \leqq A_0\int_{\frac{4R^2}{t}}^{\infty}\frac{1}{\sqrt{y}}e^{-y}dy = \gamma(R),$$

where A_0 is an absolute constant. Taking R sufficiently large so that:

(I.6) $$0 < \gamma(R) < \frac{1}{2},$$

we have

(I.7) $$|v|_{C[0,T]}$$

$$\leq \frac{2}{1-2\gamma(R)} \{ \left| \int_0^t G(R,t\,;\,-R,\tau)\dot{f}(\tau)d\tau \right|_{C[0,T]}$$

$$+ \left| \int_0^b G(R,t\,;\,\xi-R,0)\theta_0(\xi)d\xi \right|_{C[0,T]}$$

$$\equiv D(R).$$

Let $R \to +\infty$. Then $D(R) \to D(+\infty) < +\infty$. We see that for any R which satisfies (I.6), there exists an absolute constant D_0 such that:

(I.8) $$D(R) \leq D_0.$$

Thus there exists some constant D_1 such that:

(I.9) $$|v|_{C[0,T]} \leq D_1 \qquad \text{for any } R \geq b.$$

Letting $C_1(b,T) = \mathrm{Sup}\{D_1\,,\ \left|\frac{\partial\theta_1}{\partial x}(0,t)\right|_{C[0,T]}\}$,

we obtain

(I.10) $$\left|\frac{\partial\theta^\delta}{\partial x}(0,t)\right|_{C[0,T]} \leq C_1(b,T)$$

for any $R \geq b$ and any $\delta \in (0,\delta_0]$ $(\delta_0 > 0)$.

APPENDIX II

Putting $\theta - \theta' = w$, we have

$$\text{(II.1)} \qquad \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} - \frac{1}{\varepsilon}\chi_s w - \frac{1}{\varepsilon}\theta_{s'} \cdot (\chi_s - \chi_{s'}).$$

Multiplying both sides of (II.1) by $\frac{\partial w}{\partial t}$ and integrating in Q_t, we have:

$$\text{(II.2)} \qquad \int_0^t\int_0^{R_0}\left|\frac{\partial w}{\partial t}\right|^2 dx\,dt + \frac{1}{2}\int_0^{R_0}\left|\frac{\partial w}{\partial x}(x,t)\right|^2 dx$$

$$\leq \frac{C_0}{\varepsilon}\int_0^t\int_0^{R_0}|\chi_s - \chi_{s'}|\left|\frac{\partial w}{\partial t}\right| dx\,dt.$$

Here we used the monotone property of $s(t)$. By use of Schwartz's inequality, (II.2) becomes

$$\text{(II.3)} \qquad \int_0^t\int_0^{R_0}\left|\frac{\partial w}{\partial \tau}\right|^2 dx\,d\tau + \frac{1}{2}\int_0^{R_0}\left|\frac{\partial w}{\partial x}(x,t)\right|^2 dx$$

$$\leq \frac{C_0}{3}\sqrt{T}\,|s - s'|^{1/2}_{C[0,T]}\left|\frac{\partial w}{\partial t}\right|_{L^2(Q_T)}$$

for any $t \in (0,T)$.

From (II.3) we have

$$\text{(II.4)} \qquad \left|\frac{\partial w}{\partial t}\right|_{L^2(Q_T)} \leq \frac{C_0}{\varepsilon}\sqrt{T}\,|s - s'|^{1/2}_{C[0,T]}$$

By (II.3) and (II.4), we have:

$$\text{(II.5)} \qquad \int_0^{R_0}\left|\frac{\partial w}{\partial x}(x,t)\right|^2 dx \leq 2\left(\frac{C_0}{\varepsilon}\right)^2 T\,|s - s'|_{C[0,T]}.$$

On the other hand, we have

(II.6) $$|w(x,t)| \leq \int_0^x \left|\frac{\partial w}{\partial x}\right| dx \leq \sqrt{R_0} \sqrt{\int_0^{R_0} \left|\frac{\partial w}{\partial x}\right|^2 dx}$$

$$\leq \sqrt{2R_0 T} \cdot \frac{C_0}{\varepsilon} |s - s'|^{1/2}_{C[0,T]} .$$

(II.6) implies (4.18).

APPENDIX III

Putting $y_1 - y_2 = W$, we have

$$\begin{cases} \dfrac{\partial W}{\partial t} = \dfrac{\partial^2 W}{\partial x^2} + \dfrac{1}{\varepsilon}(y_1+\varepsilon a_1)^- - \dfrac{1}{\varepsilon}(y_2+\varepsilon a_2)^- + a_1 - a_2 & \text{in } Q_T, \\ \dfrac{\partial W}{\partial x}(0,t) = 0 & 0 < t < T, \\ W(B_0,t) = 0 & 0 < t < T, \\ W(x,0) = 0 & 0 < x < R_0. \end{cases}$$

Here $a_i = -\dfrac{ds_i}{dt} + h$ $(i=1,2)$. Let $U(x,\xi\,;\,t\,;\,\tau)$ be the Green's function of the linear system;

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & \text{in } Q_T, \\ \dfrac{\partial u}{\partial x}(0,t) = 0 & 0 < t < T, \\ u(R_0,t) = 0 & 0 < t < T. \end{cases}$$

Then we have

$$W(x,t) = \int_0^t d\tau \int_0^{R_0} U(x,t\,;\,\xi,\tau)\{\frac{1}{\varepsilon}(y_1+\varepsilon a_1)^- - \frac{1}{\varepsilon}(y_2+\varepsilon a_2)^- + a_1 - a_2\}d\xi.$$

By (4.29), we have

$$|W(x,t)| \le \int_0^T d\tau \int_0^{R_0} U(x,t;\xi,\tau)\{\frac{1}{\varepsilon}|y_1-y_2| + 2|a_1-a_2|\}d\xi. \tag{III.1}$$

Noting that

$$(\text{III}.2) \qquad \int_0^{R_0} U(x,t\,;\,\xi,\tau)\,d\xi = 1,$$

we have

$$(\text{III}.3) \qquad |W|_{C(\bar{Q}_T)} \leqq \int_0^T \{\frac{1}{\varepsilon}|y_1 - y_2|_{C(\bar{Q}_T)}$$

$$+ 2\left|\frac{ds_1}{dt} - \frac{ds_2}{dt}\right|_{C[0,T]}\}dt.$$

REFERENCES

[1] R. Glowinski, J.L. Lions and R. Trémolières: Analyse numérique des Inéquations variationelles, (Dunod, Gauthier-Villars, 1976)

[2] A. Friedman: Free boundary problems for parabolic equations I. Melting of solids, (J. Math. Mech. 8, 1959)

[3] A. Friedman and D. Kinderlehrer: A class of parabolic quasivariational inequalities, (J. of Diff. Eq. 21)

[4] H. Fujita and N. Sauer: Construction of weak solutions of the Navier-Stokes equation in a noncylindrical domain, (Bull. Amer. Math. Soc. 75, 1969)

[5] H. Kawarada and M. Natori: On numerical methods for the Stefan problem by means of the finite difference and the penalty, Functional analysis and numerical analysis (Proc. Japan-France seminar, Tokyo-Kyoto, 1976)

[6] H. Kawarada: Numerical methods for free surface problems by means of penalty, (Lecture Notes in Mathematics, Springer-Verlag, 704, 1979)

[7] O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Uralćeva: Linear and quasi-linear equations of parabolic type, ("Nauka", Moscow, 1967; English transl., Math. Monographs, vol.23, Amer. Math. Soc., Providence, R.I., 1968)

[8] M. Larrecq, C. Saguez and M. Wanin: Modèle mathématique de la solidification en coulée continue tenant compte de la convection à l'interface solide-liquide, (Revue francaise de Métallurgie, 1978)

[9] J.L. Lions: Une remarque sur les problèmes d'évolution non linéaires dans des domaines non cylindriques, (Rev. Roumaines Math. Pures Appl., 9, 1964)

[10] Nguyen-Dinh-Tri: Une inégalité quasi-variationelle issue d'un problème a frontière libre pur l'équation de la chaleur, (Ecole Polytechnique, 1978)

[11] C. Saguez: Un problème de Stéfan avec source sur la frontière libre, (Rapport Laboria, 268, 1977)

[12] M. Sermange: Une méthode numérique en bifurcation application à un problème à frontiere libre de la physique des plasmas, (Appl. Math. and Optim. 5, 1979)

[13] I. Yanagisawa: Numerical analysis for free boundary problem in conduction of heat using penalty and finite difference methods, (Annual Reports of the 31th Scientific Lectures of JSCE, Part 3, 1976)

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As a typical free boundary problem, a Stefan problem is studied from two analytical and numerical points of view. In the first one, by changing the dependent variable which stands for the temperature distribution, the Stefan problem is transformed into a variational inequality (V.I.). It is well known that V.I. can be approximated by a penalized problem. The second one is the method of the integrated penalty which gives a new interpretation of this penalized problem. For these different problems, existence and convergence theorems are given and, moreover, numerical methods to solve them are presented. Finally some numerical results are given.

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